MATH 22

Lecture J: 10/2/2003

MORE SETS, & PROBABILITY

A wise gamester ought to take the
dice even as they fall, and pay quietly,
rather than grumble at his luck.

—Sophocles

Administrivia

- http://denenberg.com/LectureJ.pdf
  (You really should be looking at these notes!)

- Any further questions on the tests: Office hours today!

- Project 4 will be handed out Tuesday, due 10/14

- It's too late to drop!

Brief Review

- Set, element of a set
- Subsets and proper subsets
- Set equality
- The empty set
- Set cardinality
- Power set
- Union and intersection; disjoint sets
- Zillions of laws about union and intersection
- Relative complements
- Universes and absolute complements
- Laws of complement
- Symmetric difference
- Duality [lightly]
- Venn diagrams
- Index sets [lightly]
- Examples of proofs & simplifications

Membership Tables

If you liked truth tables, you'll like membership tables.

A *membership table* characterizes expressions involving sets and set operators. Suppose we have the expression

$$S = (A \,\square\, (B - C)) \,\square\, (C \,\square\, (-B))$$

There are eight possibilities for an element of the universe: It can be in A but not B or C, in A and B but not C, in none of the three, in all of the three, etc. The membership table for S has a row for each possibility, and for each row calculates whether an element with that row's membership properties is in S. [see blackboard]

What can we do with a membership table? Suppose we make a membership table for another expression T and the final column for T is the same as the column for S. Then we can conclude that $S = T$, because any element x must appear in one of the rows of the table, and that row tells us either that x is in both S and T or neither of them. So there can't be any element that's in S but not T or T but not S, that is, S and T are equal.

[blackboard example of $(A \,\square\, C) \,\square\, (A - B)$]

Inclusion / Exclusion 2

Venn diagrams help us prove certain counting theorems.

Suppose we know that 1300 Tufts students pierce their left ear, and 1700 Tufts students pierce their right ear. How many Tufts students have pierced ears? Answer: We don't have enough information to tell! All we know is that it can't be more that 3000 or less than 1700.

The crucial piece of information is this: How many students have *both* ears pierced? If we know this, we can calculate the answer. E.g., if we know that 800 have both ears pierced, then there are 2200 with pierced ears.

Expressed set-theoretically, we're using this fact:

$$|A \cup B| = |A| + |B| - |A \cap B|$$

[proof by Venn diagram, on the blackboard]

Here's what's happening: If you just add together $|A|$ and $|B|$, you're *double counting* anything that's in both; you correct for this by subtracting the number that were double counted, namely, the things in $A \cap B$.
Only when A and B are disjoint ($A \cap B = \varnothing$) can you just add their cardinalities to get the cardinality of the union.

Inclusion / Exclusion 3

We can extend this trick to three sets. We get:

$$|A \cup B \cup C| =$$

$$|A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

[again, proof by Venn diagram]

In words: If we just add *A*, *B*, and *C* together we double count some stuff and *triple* count some stuff! Subtracting the three pairwise intersections corrects for the stuff that's double counted, but *overcorrects* for the stuff that's triple counted because we subtract it *three* times instead of twice. We must add back in the stuff that's triple counted so that it gets counted exactly once.

Example: Suppose the dessert choices at the all-you-can-eat café are apple, mince, and cherry pie. Suppose 100 people had apple, 120 mince, and 160 cherry. There's no way to tell how many had a dessert. But if we know how many had apple & mince, apple & cherry, cherry & mince, and all three, we can figure it out.

Generalized to *N* sets, this idea becomes the beautiful *Principle of Inclusion and Exclusion*. Grimaldi 8.1.

Ordered Tuples

Intuitive Definition: An *ordered pair* of objects is a pair of objects put together in order. If a and b are any two objects, we write (a,b) to denote the ordered pair with first component a and second component b. (An ordered pair is kind of like a set of two things, but with order.)

Functional Definition: Two ordered pairs (a,b) and (c,d) are *equal* if $a=c$ and $b=d$.

Note that (a,b) and (b,a) are *not* equal, unless $a=b$.

Be sure you understand the differences between the ordered pair (a,b) and the set $\{a,b\}$. E.g., $\{a,a\} = \{a\}$, but there's no analogous statement about (a,a).

We can extend the concept of ordered pairs to ordered triples (a,b,c), ordered quadruples, and in general ordered n-tuples $(a_1, a_2, a_3, \ldots, a_n)$.

For the curious only: If we were doing this rigorously, we'd define (a,b) as $\{ \{a, b\} , \{ a \} \}$. You might be amused to verify that with this definition ordered pairs have the properties described above.

Cartesian Product

With ordered pairs understood, we can tackle another set operation.

Definition: Let A and B be any two sets. The *Cartesian product* of A and B, written $A \times B$, is the set of all ordered pairs in which the first component is a member of A and the second component is a member of B. In symbols:

$$A \times B = \{ (x,y) \mid x \in A \text{ and } y \in B \}$$

Example: Let $A = \{ 1, 8 \}$ and $B = \{ 2, 8, 9 \}$. Then

$$A \times B = \{ (1,2), (1,8), (1,9), (8,2), (8,8), (8,9) \}$$

Example: $\underline{\boldsymbol{Z}} \times \underline{\boldsymbol{R}} = \{ (3, 4.2), (-999, \square), (8, 8), \ldots \}$

Elementary facts:

$A \times B$ does *not* equal $B \times A$ (unless $A = B$)

$(A \times B) \times C$ does *not* equal $A \times (B \times C)$

$| A \times B | = (| A |)(| B |)$

For any set S, $S \times \varnothing = \varnothing \times S = \varnothing$

Intro to Probability

To calculate how likely it is that something will happen, count the number of ways it can happen, and divide by the total number of ways that anything at all can happen.

Example: How likely is it that, if you toss a coin, it will land heads? That can happen in only 1 way, and there are two things that might happen, so the answer is 1/2.

Example: How likely is it that, if you pick a letter from the alphabet, you'll get a vowel? There are five ways this can happen (aeiou) and there are 26 ways that anything can happen (a–z). Dividing, the answer is 5/26.

Example: How likely is it that, if you draw five cards at random from a deck, you'll five cards all of the same suit? The number of ways of getting such a hand is 4C(13,5) [why?] and the number of ways of drawing five cards is C(52,5) as you painfully remember. So the probability of a flush is 4C(13,5) / C(52,5).

This is all the deeper we're going in probability for now. The next few slides say nothing more! We just do it a bit more carefully and with the standard terminology.

Experiments & Outcomes

Suppose we do something that might turn out in lots of different ways, but we don't know which will happen. Examples: flip a coin, toss a die, toss a hundred dice, pick a card from a deck. The thing we do is called an *experiment* (e.g., pick a card) and the result is called the *outcome* of the experiment (e.g., the ace of clubs).

The set of all outcomes of an experiment is called the *sample space* of the experiment and is usually denoted ***S***. Examples:

Experiment: Flip a coin. ***S*** = { heads, tails }

Experiment: Toss a die. ***S*** = { 1, 2, 3, 4, 5, 6 }

Experiment: Toss 2 dice. ***S*** = { 2, 3, 4, 5, . . . , 11, 12 }

Experiment: Toss 2 dice. ***S*** = { (1,1), (1,2), . . . , (6,6) }

Experiment: Buy a lottery ticket. ***S*** = { win, lose }

We prefer sample spaces where each outcome is "equally likely", a concept that we can't define (but we know it when we see it!). Such sample spaces are the useful ones. The third and fifth sample spaces above do *not* have this property. It's sometimes hard to select the sample space appropriately.

Probability

Assume that we have an experiment and a *finite* sample space ***S*** of equally likely outcomes. We define the *probability* of an outcome *x*, written Pr(*x*), as 1 / | ***S*** |.

For example, if we toss a coin, Pr(heads) = 1/2. If we roll a die, Pr(3) = 1/6. If we roll 2 dice, Pr((2,5)) = 1/36. (If we buy a lottery ticket, Pr(win) is *not* 1/2, because we've got a bad sample space!)

A subset of the sample space, that is, a set of possible outcomes of the experiment, is called an *event*. E.g.:

Toss a die. The event "even" is the set { 2, 4, 6 }.

Toss 2 dice. The event "total 8" is the set

{ (2,6), (6,2), (4,4), (5,3), (3,5) }

Pick a card from a deck. The event "jack" is the set

{ jack of □, jack of , jack of , jack of }

Toss a die. The event "3" is the set { 3 }. (An event can contain only a single outcome!)

Toss 4 dice. The event "total 30" is the empty set. (An event can have zero outcomes!)

Toss a die. { 1, 5 } is an event. (Any subset is an event.)

Pr(Event)

Suppose we have an experiment with sample space $\underline{\boldsymbol{S}}$. If E is an event, that is, $E \subseteq \underline{\boldsymbol{S}}$, we say that the *probability of E*, written Pr(E), is $|E| \;/\; |\underline{\boldsymbol{S}}|$.

This is exactly and only what we said a few slides back! You count the number of outcomes in the event you're interested in (e.g., the number of flushes), and divide by the total number of outcomes in the sample space (e.g., the number of five-card hands). Assuming that the outcomes in the sample space are equally likely, the result is the probability of the event.

(Now you see why we often study probability in conjunction with counting!)

The whole of section 3.4 is nothing but examples of this technique. Define the experiment, define a sample space of equally likely outcomes, find the cardinality of the sample space ($|\underline{\boldsymbol{S}}|$, the denominator), find the cardinality of some event ($|E|$, the numerator), and divide to find the probability of event E.

[Fuzzy Sets]

In our discussion of sets, an element is either a member of a set or not. There's another notion of sets were membership isn't so strictly yes or no.

A *fuzzy set* is a set to which elements belongs *to some degree*; that is, the boundaries of the set are not sharp, as in traditional sets. Given an object x, it can belong to a fuzzy set F with any real value from 0 to 1.

Example: Let T be the fuzzy set of tall people. A typical basketball player might be 99% in T. Larry might be 80% in T. Napoleon was 0% in T.

Note that this doesn't mean "80% of Larry belongs to T". It means "Larry 80% belongs to T", or "the degree to which Larry belongs to T is 80%".

In fuzzy sets, union and complement turn into max and min. So if x 80% belongs to F_1 and 65% belongs to F_2, then x 80% belongs to $F_1 \cup F_2$, and x 65% belongs to $F_1 \cap F_2$.

With fuzzy sets, we can avoid having to establish a cutoff value for "tall" in order to have a set of tall people.