MATH 22

Lecture U: 11/13/2003

GRAPHS

[A man who] doth rebate and blunt his natural edge / With profits of the mind, study and fast.

—*Measure for Measure*, I:4

The beauty of the world has two edges, one of laughter, one of anguish, cutting the heart asunder.

—Virginia Woolf,
A Room of One's Own

Administrivia

- http://denenberg.com/LectureU.pdf

- The Grading Scale (pretend we're back at Lecture A):

[98, 100]	A+
[92.5, 98)	A
[89.5, 92.5)	A–
[86.5, 89.5)	B+
[82.5, 86.5)	B
[79.5, 82.5)	B–
[76.5, 79.5)	C+
[72.5, 76.5)	C
[69.5, 72.5)	C–
[66.5, 69.5)	D+
[62.5, 66.5)	D
[59.5, 62.5)	D–
[0, 59.5)	F

Today: Graphs, a supremely important tool of discrete mathematics. (Another good lecture to take notes, since lots of stuff will be on the blackboard.) At first it will be a little heavy on definitions and a little light on hard results, skipping G's examples.

Graphs

A *graph* is a nonempty set of dots connected by lines.

The dots are called ***vertices***, or ***nodes***, or *points*.
The lines are called ***edges***, or *arcs*.
Edges are said to be *incident* on the nodes they join.
Nodes connected by an edge are *adjacent*, or *neighbors*.

In a *directed graph* the edges have arrowheads on one end; an edge starting at node X and ending at node Y differs from an edge starting at Y and ending at X, and the graph may contain either independently, or both.
In an *undirected graph* edges have no arrowheads.

An edge, directed or undirected, from a node X back to itself is called a *self-loop*. We often restrict ourselves to loop-free graphs, especially in the undirected case.

If "copies" of edges are permitted (e.g., two edges from X to Y in a directed graph, or two edges between X and Y in an undirected graph), it's a *multigraph*. If edges can be incident on more than two nodes, it's a *hypergraph*.
We usually assume that graphs are not multigraphs and we never (in this course) consider hypergraphs.

Walk, Trail, Path, Cycle

If you start at a node, follow an edge to another node, follow another edge to another node, etc., you have traced a *walk* of the graph:

$$x_0, e_1, x_1, e_2, x_2, e_3, \ldots x_{n-1}, e_n, x_n$$

(The x_i are nodes and the e_i are edges.)

If the graph is directed, the edges must be traversed in the direction of the arrowheads. But nodes can be revisited and edges can be reused any number of times.

The *length* of the walk is the number of edges (here n).

If $x_0 \neq x_n$, the walk is called *open*. If $x_0 = x_n$ and $n > 0$, the walk is called *closed*.

If no edge is repeated in the walk, the walk is called a *trail*. If in addition the walk is closed, it's a *circuit*.

If no vertex is repeated in the walk, the walk is called *simple*, or a *path*. If in addition the walk is closed and $n > 2$, the walk is a *cycle*. A graph with no cycles is called *acyclic*.

Theorem: If there is a walk from X to Y, there is both a trail and a simple path from X to Y.

Connected Graphs

An undirected graph is called *connected* if there is a path between any two of its nodes.

A directed graph is called *connected* if the graph produced by erasing the arrowheads is connected.

If, in a directed graph, there is a path between any two nodes even without erasing the arrowheads, the graph is called *strongly connected*. (This term has no meaning for undirected graphs.)

Consider only undirected graphs for the moment. The "separated pieces" of any graph G are called the *connected components* of G. Each connected component consists of a subset of the nodes and edges of G that form a connected graph.

The number of connected components of a graph G is written $\square(G)$. G is connected if and only if $\square(G) = 1$.

Directed graphs have *strongly connected components*, which are much much harder to see.

Formal Definition

A *directed graph* is an ordered pair (*V*,*E*) where *V* is a finite, nonempty* set whose elements are called *vertices*, and where *E* is a (possibly empty) set of ordered pairs of vertices, called *edges*. Note that this definition permits self-loops, but not multigraphs. Note also that the elements of *V* can be anything at all.

An *undirected graph* is an ordered pair (*V*,*E*) where *V* is as above and *E* is a set of subsets of *V* each of which has cardinality 2. This definition does *not* permit self-loops.

Examples:

({u, v, w, x, y, z},
{ (v,u), (w,u), (u,w), (w,y), (y,v), (x,x), (y,y) })

({u, v, w, x, y, z} ,
{{u,v}, {w,u}, {y,x}, {y,v}, {w,v}, {y,z}, {v,z}, {x,z}})

*Some permit graphs in which $V=\emptyset$, but we consider such graphs to be pointless. [Harary]

Subgraphs

Let $G = (V,E)$ be a graph. Let V_1 be a nonempty subset of V and let E_1 be a (possibly empty) subset of E such that $G_1 = (V_1, E_1)$ is a graph. Then G_1 is called a *subgraph* of G. [Blackboard examples]

Note that this definition is equally good for directed or undirected graphs. By the way, how could $E_1 \subseteq E$ be otherwise in this definition?

Here is a digression just to fill up the slide:

Theorem: Let G be an undirected graph with n vertices. The following conditions on G are equivalent:

- Any two nodes of G are joined by a unique simple path.
- G is both connected and acyclic.
- G is connected and has n–1 edges.
- G is acyclic and has n–1 edges.
- G is connected, but deleting any edge disconnects it.
- G is acyclic, but adding any edge creates a cycle.

An undirected graph that satisfies any of these conditions, and hence all of them, is called a *tree*.
(Trees are themselves a huge topic, which we skip, alas.)

Special Subgraphs

If $G = (V,E)$ is a graph, a *spanning subgraph* of G is any graph (V,E_1) where $E_1 \subseteq E$. That is, a spanning subgraph of G is any subgraph of G that has the same vertices as G. That is, a spanning subgraph of G is just G with some edges erased.

Theorem: Graph $G = (V,E)$ has $2^{|E|}$ spanning subgraphs.

Given a graph $G = (V,E)$ and a nonempty subset V_1 of V, the *subgraph of G induced by* V_1 is the graph (V_1, E_1) where E_1 is the subset of E consisting of those edges that are incident only on nodes of V_1. [Blackboard example]

Said another way: The subgraph of $G = (V,E)$ induced by $V_1 \subseteq V$ is the subgraph obtained by erasing from G those nodes not in V_1, and erasing *only* edges of G that *must* be erased because they no longer touch two nodes. (That is, don't erase any edges you don't have to erase.)

Let $G = (V,E)$ be a graph. If $v \in V$, we write $G - \{v\}$ to denote the subgraph of G induced by $V - \{v\}$.
If $e \in E$, we write $G - \{e\}$ to mean G with edge e erased (is this an induced subgraph?). We can similarly write things like $G - \{v_1, v_2, v_3\}$ or $G - \{e_1, e_2\}$.

Complete Graphs/Degree

A *complete undirected graph on n vertices* is a graph (V,E) where $|V| = n$ and $\{v_1, v_2\} \in E$ for every pair of distinct nodes $v_1, v_2 \in V$.

Said another way: A complete graph on n vertices is a graph with n vertices and every possible edge, which is to say, $n(n-1)/2$ edges [why?]. We write K_n to denote a "generic" complete undirected graph on n vertices.
[Blackboard drawing: K_1, K_2, K_3, K_4, K_5, maybe K_6.]

If v is a vertex of an undirected graph, the *degree of v*, written *deg(v),* is the number of edges incident on v.
In a complete graph K_n each vertex has degree $n-1$ [NOT n!]. An *isolated vertex* of a graph is a vertex with degree 0. (In a directed graph we speak of the in-degree and the out-degree of v, with obvious definitions.)

Theorem: In any undirected graph G, the sum of deg(v) over all vertices in G is twice the number of edges of G.

A graph in which all vertices have the same degree is called *regular*. All complete graphs are regular.

Graph Complements

Let $G = (V,E)$ be an undirected graph. The *complement of G*, written $–G$ here but usually with a bar *over* the G, is the graph (V,E_1) where, for any distinct $v_1, v_2 \in V$, we have $\{v_1, v_2\} \in E_1$ if and only if $\{v_1, v_2\} \notin E$.

More simply: the complement of a graph G is a graph with the same vertices, and with an edges exactly where G does not have edges. [Blackboard examples]

Fact: If G has n vertices and k edges, its complement has exactly $n(n–1)/2 – k$ edges.

Fact: If G has n vertices and is regular of degree k, then its complement is regular of degree $n–k–1$.

If G is connected, $–G$ may or may not be connected, but if G is disconnected then $–G$ is necessarily connected!

Proof: Suppose G is disconnected and let v and w be any two vertices of G. If v and w are in different connected components of G then there's an edge between them in $–G$. If v and w are in the same component, pick any node z in a different component; there's a path from v to w in –G by going from v to z to w.

Graph Isomorphism

Suppose G_1 is an undirected graph with vertex set $V = \{1, 2, 3, 4, 5, 6\}$ and edges $\{1,5\}$, $\{2,6\}$, and $\{3,6\}$. (Vertex 4 is an isolated vertex of this graph.)

Now let G_2 be a graph with vertices $\{A,B,C,D,E,F\}$ and edges $\{E,B\}$, $\{A,F\}$, and $\{D,A\}$. Clearly G_1 and G_2 aren't "equal" since their vertex sets aren't equal. But just as clearly they are "the same" in some sense. How can we make this precise?

Definition: Undirected graphs (V_1,E_1) and (V_2, E_2) are called *isomorphic* if there is a bijection $f : V_1 \rightarrow V_2$ that *preserves adjacency*, that is, such that $\{v_1, v_2\} \in E_2$ if and only if $\{f(v_1), f(v_2)\} \in E_2$.

That is, graphs are isomorphic if there is some way of putting their vertices in one-to-one correspondence so that the edges line up as well. In the example above,
$f(1) = E$, $f(2) = F$, $f(3) = D$, $f(4) = C$, $f(5) = B$, $f(6) = A$
is an appropriate bijection. (There are other appropriate bijections, and many bijections that aren't approriate, but this doesn't matter; the graphs are isomorphic if there is at least one such bijection.)

More on Isomorphism

If G_1 and G_2 are isomorphic, then they must have the same number of vertices and the same number of edges. Moreover, if G_1 has k vertices of degree d, then G_2 must also have k vertices of degree d. And any two isomorphic graphs have the same number of connected components.

All these things are easy to check, but can only provide a negative answer to the question. (E.g., if two graphs don't have the same number of edges, they're definitely not isomorphic.) But two graphs can have the same number of nodes, edges, components, matching degrees, etc., and still not be isomorphic. [Blackboard examples]

Theorem: If two graphs are both complete graphs on n vertices, they are isomorphic. If two graphs both have n vertices and zero edges, they are isomorphic. If two graphs both have n vertices and a single edge, they are isomorphic.

Theorem: If two graphs are isomorphic, then so are their complements. Moreover, the same bijection demonstrates the isomorphism.

Eulerian Circuits

Suppose G is an undirected graph (which can be a multigraph). A circuit that traverses every edge of G and visits every node of G is called an *Eulerian circuit*. A graph that has an Eulerian circuit is called *Eulerian*. [Blackboard picture of the bridges of Königsberg.]

(A circuit that traverses every edge of G must visit every node unless there are isolated nodes, in which case it can't visit every node. So an Eulerian graph has no isolated nodes. Note that a node may be visited any number of times in the course of an Eulerian circuit.)

Theorem (Euler, 1736): An undirected graph is Eulerian if and only if it is connected and has no vertices with odd degree.

Proof (only if): Suppose G has an Eulerian circuit starting and ending at node v. This circuit visits all nodes of G, so clearly G is connected. Each time a node is visited, we go in one edge and out another, accounting for 2 edges in the degree; this shows that all nodes, except perhaps v, have even degree. To show it of v, simply start the circuit in another place!

Euler's Proof, cont.

(If) We assume now that G is connected and has vertices of even degree, and we need to show that G is Eulerian.

Pick any vertex v as a starting point and construct a walk W that doesn't reuse edges. (You can certainly get started, since G has no isolated vertices!) As you walk, you can leave any node that you enter, since each vertex has even degree. Therefore, the walk stops only when it returns to v and no unused edges out of v are left.

If W contains all edges of G, we're done. Otherwise, construct a subgraph G_1 of G by removing from G all edges in W and all vertices that become isolated. Though G_1 may not be connected, each of its nodes has even degree. Also, since G is connected, some node v_1 of G_1 must lie on walk W. So construct a new walk W_1 starting at v_1 as before; W_1 must end eventually at v_1. Splice W_1 onto W.

If now W contains all edges of G, we're done. Otherwise we can repeat the process until we have no edges left in the shrinking subgraph. QED

Comments on Euler

An *Eulerian path* in an undirected graph is one that traverses every edge and visits every node, but isn't necessarily a circuit.

If a connected undirected graph with no isolated nodes has exactly two vertices of odd degree, we can carry out Euler's construction starting at one of these vertices. The first walk W will necessarily end at the other! But after that, all vertices of the shrinking subgraph have even degree, and the rest of the construction doesn't chang—it still splices everything into W. This proves that any connected undirected graph with exactly two odd vertices has an Eulerian path.

A path in an undirected graph is called *Hamiltonian* if it visits every vertex, and an undirected graph is called Hamiltonian if it contains a Hamiltonian path. *There is no efficient way to determine whether a given graph is Hamiltonian*.