MATH 22

Lecture Z: 12/4/2003

FINAL REVIEW

O passi graviora, . . . forsan et haec
olim meminisse iuvabit.
—*Aeneid,* I. 199

What a long, strange trip it's been.
—Jerry Garcia, "Truckin'"

Administrivia

- http://larry.denenberg.com/math22/LectureZ.pdf

- FINAL EXAM THURSDAY DECEMBER 11, 8:30 – 10:30 AM, ROBINSON 253 (same as Exam 3)

- Office hours during reading period: After class tonight (as usual), then Friday through Wednesday morning until 10 AM, *but only by prearrangement*; if you want to consult you must call or write! Email answered day or night. 617-995-1234 days. (Any desire for a review session, e.g. Tuesday 4 PM?)

- Office hours after the exam: Monday 12/15 and Wednesday 12/17 from 9:00 to 10:00 AM; other times too, all by prearrangement.

- Dr. Elder's office hours next week: Tuesday & Wednesday 10 AM to noon, room 208. Open to all.

> O you who have suffered . . . perhaps someday even these things will be pleasant to remember.
>
> —Virgil, *Aeneid* I. 199

Final Topic: Infinity

Let A and B be sets. If there is a bijection between A and B then we say that A and B *have the same cardinality* and we write $|A| = |B|$. Note that this definition is valid for any two sets whatever.

We call a set A *infinite* if there is a **proper** subset B of A such that $|A| = |B|$. If a set is not infinite then it is *finite*.

Theorem: $\underline{\boldsymbol{Z}}$, the set of integers, is infinite.

Proof: $f(x) = 2x$ is a bijection between the set of integers and the set of even integers. This also proves that there are the same number of even integers as there are integers. The same statement applies to primes, squares, cubes, multiples of 3, multiples of 1000000, etc.

We call a set *countably infinite* if it has the same cardinality as $\underline{\boldsymbol{Z}}$. A set is *countable* or *denumerable* if it is finite or countably infinite, and *uncountable* otherwise.

Theorem: $\underline{\boldsymbol{Q}}$, the set of rational numbers, is countable.
Theorem: $\underline{\boldsymbol{R}}$, the set of real numbers, is uncountable.
Theorem: If S is any set, then 2^S, the set of all subsets of S, does not have the same cardinality as S. (So given any size of infinity there is always a larger infinity.)

Cantor's beautiful proofs, using a proof technique called *diagonalization*, omitted for lack of space and time.

What You Need

I. Counting
permutations, combinations, with & without replacement, Catalan numbers, binomial theorem

II. Logic
connectives, truth tables, “laws”, quantifiers, proofs, rules of inference, proof by contradiction, proof by contrapositive

III. Sets
elements & subsets, equality, set operations, power set, Venn diagrams, ordered pairs, Cartesian product, probability, pigeonhole principle

IV. Mathematical Induction (weak and strong form)

V. Functions
domain, codomain, range, image, preimage, injections, surjections, bijections, floor, ceiling, composition, inverses of functions, growth rates, O notation, algorithm analysis

What You Need, cont.

VI. Relations

properties, equivalence relations and partitions, partial orders and Hasse diagrams

VII. Number Theory

divisibility, primes, composites, division theorem, base conversion, gcd, Euclidean algorithm, Fundamental Theorem of Arithmetic

VIII. Graphs

directed & undirected, node degree, edge counting, connectivity, complement, subgraphs, isomorphism, Eulerian circuits, Euler's Theorem, Hamiltonian paths, planar graphs, Euler's Formula, vertex coloring, chromatic number, 4CT

The Official Word on the Exam:

"The exam will be 10 questions, similar difficulty to the previous three. You should study what you missed on the previous exams, look over the projects, and especially all the homework questions. All sections covered by the homeworks are fair game. Emphasis on proofs, especially by induction."

True/False/Neither?

Classify each of the following statements as true, false, or neither. For example, "1+1=2" is true, "1+1=3" is false, and "1+blue=hello" is neither since you can't add numbers and colors. All graphs are undirected.

- If a graph is planar, its chromatic number is 4.
- If a graph has chromatic number 4 or less, it's planar.
- If a node of a graph has degree ≤ 2, that node is planar.
- If a graph has 3 connected components, each with 5 nodes, each node degree 2, then the graph has 30 nodes.
- If a graph has 3 connected components, each with 10 nodes, each node degree 2, then the graph has 30 edges.
- If a graph has 3 connected components, each with 5 nodes, each node degree 3, then the graph has 30 nodes.
- $(\square x)(\square y)(\square z)\ x+y=z$
- $(\square x)(\square y)(\square z)\ x+y=z$
- $(\square x)(\square y)(\square z)\ x+y+z=w$
- Addition on the integers is symmetric, but not transitive nor reflexive, and so is not an equivalence relation.

Answers: F, F, N, F, T, T [careful!], T, F, N, N.

Graphs, planarity, …

Give an example of a graph such that

a) Both the graph and its complement are planar.

b) The graph is planar, but its complement is not planar.

c) Both the graph and its complement are nonplanar.

In each case, try to minimize the number of vertices.

Minimum number of vertices required: 1, 5, and 8.

Prove that if a graph G and its complement are both planar, then G has at most 10 vertices.

Hint: The number of edges in a planar graph is at most $3v - 6$, where v is the number of vertices.

(In fact, if G and its complement are both planar then G has at most 8 nodes, but this is harder to prove.) Find a planar graph with 8 nodes whose complement is planar.

Show that K_4 is self-dual. (Draw it and check!)

Consider the formula $(\square x)(\square y)(\square z)\ x+y=z$ from the previous page. How many different ways are there of arranging the quantifiers in this formula? In general, how many ways are there of quantifying n variables?

Answers: $(3!)(2^3)$ and, in general, $(n!)2^n$.

Logic, relations, …

Prove that the relation $\Rightarrow$ on the set of truth values {true,false} is a partial order. Draw its Hasse diagram.

Reflexive: T$\Rightarrow$T and F$\Rightarrow$F. Symmetric: If $x \Rightarrow y$ and $y \Rightarrow x$ then x and y can't differ; one or the other would be T$\Rightarrow$F! Transitive: [Proof by contradiction omitted.] The Hasse diagram has two points, with T on top.

We can also think of $\Rightarrow$ as a relation on the set of all logical formulas. Is this relation a partial order?

No. If F and G are logical formulas, we can have both $F \Rightarrow G$ and $G \Rightarrow F$ without $F = G$ (for example, let F be pq and let G be qp). So $\Rightarrow$ on formulas is not antisymmetric.

Explain why the relation $\equiv$ on the set of all logical formulas is an equivalence relation. Give three members of the equivalence class of the formula '$p \quad q$'.

If F, G, and H are any logical formulas, then the laws of logic say that $F \equiv F$, that if $F \equiv G$ then $G \equiv F$, and that if $F \equiv G$ and $G \equiv H$ then $F \equiv H$; this shows that $\equiv$ is an equivalence relation. $[p \quad q]$ is the set of all logical formulas equivalent to $p \quad q$. Some members of this set are $p \quad q \quad q$, $p \quad q \quad p$, $p \quad q \quad q(\neg q)$, and a zillion others, including of course $p \quad q$ itself.

Counting, GCD, . . .

Suppose finite set S has n elements, one of which is x. How many subsets of S contain x? How many ways are there to choose m elements of S such that x is chosen?

The answer to the first part is 2^{n-1}. For the second part, we need to choose m elements for a subset, but since we must choose x we have only $m-1$ choices remaining. Also, we have only $n-1$ elements to choose from, since x can't be one of the choices. Answer: $C(n-1, m-1)$.

Suppose that a and b are relatively prime. Prove that $\gcd(a+b, a-b)$ is either 1 or 2.

Let x be the gcd. Since x divides both $a+b$ and $a-b$ it divides their sum, which is $2a$. And if $x \mid 2a$, then either $x \mid 2$ or $x \mid a$. (Note: This last statement is true only because 2 is prime!) If $x \mid 2$ then clearly x is 1 or 2, so all that's left is the case $x \mid a$.

We can repeat this reasoning using the *difference* of $a+b$ and $a-b$ rather than their sum; this difference is $2b$. We get that either $x \mid 2$ or $x \mid b$. Again, we're done if $x \mid 2$, so we're left with the case $x \mid b$.

But if $x \mid a$ and $x \mid b$ then $x = 1$; this is the definition of "relatively prime". So in this case too we're done.

Final Problem

You are given 48 circles arranged in 8 rows, 6 circles in each row. In how many ways can you choose 8 circles, one from each row, in such a way that you choose the *leftmost* circle in each row?